Tutorial 7 : Completion Theorem

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Completion Theorem

Def A completion of a metric space (X,d) is a pair $((Y,\rho),\Phi)$, where

① (Y,ρ) is a complete metric space (i.e. every Cauchy sequence converge)

② $\Phi: X \to Y$ is an isometric embedding, i.e. for any $x, x' \in X$,

$d(x,x') = \rho(\Phi(x), \Phi(x'))$

③ $\overline{\Phi(X)} = Y$

Thm Every metric space (X,d) has a completion.

Proof (Without using Cauchy completion as in Lecture 3, p.25-27)

Step 1: Consider the normed space of real-valued bounded continuous functions endowed with sup norm $(C^b(X), \|\cdot\|_\infty)$. Show that it is complete.

Step 2: Construct an isometric embedding $\Phi: X \to C^b(X)$

Step 3: Define $Y := \overline{\Phi(X)} \subseteq C^b(X)$ with induced metric $\rho(g,h) := \|g-h\|_\infty$

Show that $((Y,\rho),\Phi)$ is a completion of (X,d).

Goal Fill in the details of the above proof.

Q1) Show that $(C^b(X), \|\cdot\|_\infty)$ is complete.

Sol) Let $(f_n) \subseteq C^b(X)$ be a Cauchy sequence. Given $\varepsilon > 0$.

$\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$

$\therefore \forall x \in X$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon$

$\therefore \forall x \in X$, $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence.

By completeness of $\mathbb{R}$, $\exists!\, z_x \in \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = z_x$.

Define $f : X \to \mathbb{R}$ by $f(x) := z_x$.

We first show that f_n converges to f uniformly:

$\forall m, n \geq N$, $\forall x \in X$, $|f_n(x) - f_m(x)| < \varepsilon$

Take $m \to +\infty$: $|f_n(x) - f(x)| \leq \varepsilon$, $\forall n \geq N$, $\forall x \in X$

$\therefore \|f_n - f\|_\infty \leq \varepsilon$. Hence f_n converges to f uniformly on X.

Therefore, by the Interchange Theorem, $f \in C^b(X)$.

Also, $\lim_{n\to\infty} f_n = f$. $\therefore (f_n)$ converges in $C^b(X)$

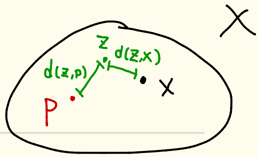
$\therefore$ Every Cauchy sequence converges. Hence, $(C^b(X), \|\cdot\|_\infty)$ is complete.

Rmk The Interchange Theorem for functions on a metric space is as follows:

Thm Let $(f_n)_{n=1}^{\infty}$ be a sequence of bounded (resp. continuous) functions on a metric space (X, d) such that (f_n) converges uniformly to f. Then f is also bounded (resp. continuous).

Pf Exercise.

(See also [Bartle-Sherbert: Introduction to Real Analysis (Fourth edition)] §8.2 Ex.7 (resp. Thm 8.2.2))



Q2) Construct an isometric embedding $\Phi : X \to C^b(X)$.

Sol) Fix $p \in X$; Define $\Phi : X \to C^b(X)$ by $\Phi(x) = f_x$, where

$f_x : X \to \mathbb{R}$ is defined as $f_x(z) := d(z,x) - d(z,p)$

Showing Φ is well-defined, i.e. $f_x \in C^b(X)$:

(i) Bounded: $\forall z \in X$, $|f_x(z)| = |d(z,x) - d(z,p)| \leq d(x,p)$.

(ii) (Lipschitz) Continuous : $\forall z, z' \in X$, $|f_x(z) - f_x(z')|$

$= |(d(z,x) - d(z,p)) - (d(z',x) - d(z',p))| \leq |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)|$

$\leq d(z,z') + d(z,z') = 2d(z,z')$.

$\therefore$ $f_x \in C^b(X)$, hence Φ is well-defined.

Showing Φ is an isometric embedding: $\forall x, x' \in X$, $\|f_x - f_{x'}\|_\infty = d(x,x')$

$[\leq]$: $\forall z \in X$, $|f_x(z) - f_{x'}(z)| = |(d(z,x) - d(z,p)) - (d(z,x') - d(z,p))|$

$= |d(z,x) - d(z,x')| \leq d(x,x')$. $\therefore$ $\|f_x - f_{x'}\|_\infty \leq d(x,x')$.

$[\geq]$: $\|f_x - f_{x'}\|_\infty \geq f_x(x') - f_{x'}(x') = d(x',x) - d(x',x') = d(x,x')$.

Hence, Φ is an isometric embedding.

Q3) Define $Y := \overline{\Phi(X)} \subseteq C^b(X)$ with induced norm $\rho := \|\cdot\|_\infty$

Show that $((Y,\rho), \Phi)$ is a completion of (X,d).

Sol) Showing $((Y,\rho), \Phi)$ is a completion of (X,d):

① (Y,ρ) is a closed subspace of $(C^b(X), \|\cdot\|_\infty)$ which is complete by Q1,

$\therefore$ (Y,ρ) is complete (By Lecture note, Ch.3, Prop. 3.1(b))

② $\Phi : X \to Y \subseteq C^b(X)$ is an isometric embedding: for any $x, x' \in X$,

$d(x,x') = \|f_x - f_{x'}\| = \rho(\Phi(x), \Phi(x'))$

③ $\overline{\Phi(X)} = Y$: follows from the definition of Y.